Hawking radiation as quantum tunneling in Rindler coordinate

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ABSTRACT: We substantiate the Hawking radiation as quantum tunneling of fields or particles crossing the horizon by using the Rindler coordinate. The thermal spectrum detected by an accelerated particle is interpreted as quantum tunneling in the Rindler spacetime. Representing the spacetime near the horizon locally as a Rindler spacetime, we find the emission rate by tunneling, which is expressed as a contour integral and gives the correct Boltzmann factor. We apply the method to non-extremal black holes such as a Schwarzschild black hole, a non-extremal Reissner-Nordström black hole, a charged Kerr black hole, a de Sitter space, and a Schwarzschild-anti de Sitter black hole.

Contents

1. Introduction

A black hole radiates thermal radiation with the Hawking temperature determined by the surface gravity at the event horizon [1]. The surface gravity is the acceleration measured at the spatial infinity that a stationary particle should undergo to withstand the gravity at the event horizon. The accelerated particle detects a thermal spectrum with the Unruh temperature out of the Minkowski vacuum [2]. The particle accelerated with the surface gravity would see the vacuum containing a thermal flux with the Hawking temperature. The thermal spectrum seen by the accelerated particle can also be understood by the interpretation that the Minkowski vacuum is restricted to a causally connected Rindler wedge due to presence of horizons just as the horizon of a black hole prevents the outer region from being causally connected with the region inside the horizon [3].

Recently Parikh and Wilczek reinterpreted the Hawking radiation as quantum tunneling [4]. Their observation is that as a particle has a negative energy just inside and a positive energy just outside the horizon, a virtual pair created near the horizon can materialize into a real pair with zero total energy, one particle on each side of the horizon. Indeed, the Hawking radiation is such particle production and can be interpreted as tunneling through the horizon. The tunneling process for particle production by geometry of a black hole is analogous to the Schwinger mechanism for pair production by an external electric field [5]. Indeed, charged pairs can be materialized from virtual pairs when the

potential energy across the Compton wavelength is comparable to the rest mass of the particle.

One approach to the tunneling interpretation of Hawking radiation is to study the tunneling motion of the s-wave of emitted radiations, which has been applied to various black holes [6 − 28] (for a review, see [17]). Another approach is to study quantum fields tunneling through the horizon in a black hole spacetime [29]. In the latter a conventional wisdom is to find the imaginary part of the Hamilton-Jacobi action, twice of which leads to a Boltzmann factor [29 − 41]. However, the emission rate by tunneling depends on the coordinate system. The radial coordinate, for instance, yields a temperature twice of the Hawking temperature. Some proposals were advanced to remedy the ambiguity. In ref. [29], the ratio of emission to absorption was used to get the correct Boltzmann factor. Also the isotropic coordinate and the proper distance from the horizon was used [35, 36], and different coordinates were tested [31]. In refs. [37] and [38] the emission rate was suggested as a contour integral, which makes the rate invariant under canonical transformations.

The purpose of this paper is to substantiate the tunneling idea within the context of quantum field theory in the black hole spacetime. The analogy between the Hawking radiation and the Schwinger mechanism has been well exploited [42 − 45]. As the Schwinger mechanism can be interpreted as the Unruh effect seen by a charged particle accelerated by the electric field, we may use the Rindler coordinate for the accelerated particle. In addition, as the Hawking radiation can be interpreted as the Unruh effect with the surface gravity, the Rindler coordinate may be used to describe the tunneling process of quantum fields. In the tunneling interpretation a local Rindler frame was first introduced in ref. [34], the Unruh effect was calculated from tunneling [39], and the discrepancy of the Hawking temperature from tunneling from the left to the right wedge and from the right to the future wedge in the full Rindler spacetime was discussed [40]. In the previous paper [41], along the lines of the Schwinger mechanism, the Rindler coordinate was used to get the correct Boltzmann factor for a charged black hole and a BTZ black hole.

Another reason for using the Rindler coordinate is that quantum tunneling of fields or particles occurs through the horizon and locally the Rindler coordinate near the horizon is an accelerated frame for the Minkowski spacetime when the acceleration is the surface gravity. Further the field equation in the Rindler spacetime has many properties in common with the equation minimally coupled with the gauge field of electric field for the Schwinger mechanism. All these imply that the Rindler coordinate is an appropriate coordinate to make full use of the analogy between the Schwinger mechanism and the Hawking radiation. We show that in the Rindler spacetime the tunneling rate of a quantum field from a causally disconnected region through the horizon explains indeed the thermal spectrum with the Unruh temperature. In fact, the tunneling rate in the Rindler spacetime takes the same form as the rate for tunneling of virtual pairs in the Schwinger mechanism.

We represent a black hole spacetime in the Rindler coordinate and then calculate the emission rate by tunneling of quantum fields through the horizon. The emission rate found in analogy with the Schwinger mechanism is given by a contour integral in the Rindler spacetime. As the contour integral is independent of coordinate transformations, the emission rate is invariant under canonical transformations. Also as quantum physics

just inside and outside horizons can be properly described by Rindler coordinates, the Rindler spacetime seems to be a natural coordinate that avoids the ambiguity from the modification of contours in changing coordinates. We get the correct Boltzmann factor for non-extremal black holes such as a Schwarzschild black hole, a non-extremal Reissner-Nordström black hole, a charged Kerr black hole, a de Sitter space, and a Schwarzschild-anti de Sitter black hole. However, our method cannot be applied to extremal black holes because there are no Rindler coordinates locally near horizons.

The organization of this paper is as follows. In section 2 we discuss some ambiguity of the Hawking radiation as tunneling. In section 3 we derive the emission rate by tunneling in the Rindler spacetime and then compare it with the Schwinger mechanism. In section 4 we use the Rindler coordinate to derive the emission rate within quantum field theory and interpret the Hawking radiation as quantum tunneling for a Schwarzschild black hole, a non-extremal Reissner-Nordström black hole, a charged Kerr black hole, a de Sitter space, and a Schwarzschild-anti de Sitter black hole. In section 5 we compare the emission rate in the Rindler coordinate with the isotropic coordinate and the proper distance from the horizon. Finally we conclude in section 6.

2. Hawking radiation as tunneling

In this section we briefly review tunneling of quantum fields through the horizon in a black hole spacetime and discuss the related problems. It was pointed out that this approach to the tunneling interpretation of Hawking radiation has ambiguities such coordinate-dependence of the Hawking temperature and non-invariance of the action under canonical transformations [37, 38].

For the sake of simplicity we consider a stationary black hole with the metric of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + h_{ij}dx^i dx^j, \tag{2.1}$$

where $h_{ij}dx^i dx^j$ is a two-dimensional metric and, for instance, becomes $r^2 d\Omega^2$ for a spherically symmetric black hole. The event horizon r_H is located at $f(r_H) = g(r_H) = 0$, near which $f(r) = f'(r_H)(r - r_H)$ and $g(r) = g'(r_H)(r - r_H)$ up to the leading term. The exceptional case of extremal black holes will be treated separately. As we are mostly concerned about the s-wave (spherically symmetric) of a massive scalar field in the black hole spacetime, we shall further restrict our investigation to the two-dimensional sector of (t, r).

The equation of the massive scalar takes the form [in units with $\hbar = c = 1$]

$$\left[\frac{1}{\sqrt{fg}} \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial r}\left(\sqrt{fg}\frac{\partial}{\partial r}\right) + m^2 \right] \Phi(t, r) = 0. \tag{2.2}$$

The action from the solution, $\Phi(t, r) = e^{iS_\pm(t,r)}$, with

$$S_\pm(t, r) = -\omega t \pm \int p(r)dr, \tag{2.3}$$

satisfies, at the leading order, the Hamilton-Jacobi equation

$$\left(\frac{\partial S_\pm}{\partial r}\right)^2 - \frac{\omega^2}{fg} + \frac{m^2}{\sqrt{fg}} = 0. \tag{2.4}$$

Here $\pm p(r) = \partial S_\pm/\partial r$ is the radial momentum of an outgoing or ingoing wave, respectively. The action is then given by

$$S_\pm(t,r) = -\omega t \pm \int_{r_0}^r \frac{dr}{\sqrt{fg}}\sqrt{\omega^2 - m^2\sqrt{fg}}. \tag{2.5}$$

There is an ultraviolet divergence from the simple pole at the event horizon, since $fg = f'(r_H)g'(r_H)(r - r_H)^2$, which contributes to an imaginary part.

To describe waves tunneling through the event horizon, r_0 is located inside the horizon. The imaginary part may be obtained by taking a semi-circle under the horizon r_H as

$$\mathrm{Im}S_\pm = \pm\frac{\pi\omega}{\sqrt{f'(r_H)g'(r_H)}}. \tag{2.6}$$

Then the amplitude square, $|\Phi|^2 = e^{-2\mathrm{Im}S_+}$, leads to the tunneling (emission) rate for the outgoing wave through the horizon:

$$P = e^{-2\mathrm{Im}S_+} = e^{-\omega/T}, \tag{2.7}$$

where

$$T = \frac{\sqrt{f'(r_H)g'(r_H)}}{2\pi}. \tag{2.8}$$

A few remarks are in order. First, note that the temperature (2.8) is the twice of the Hawking temperature

$$T_H = \frac{\sqrt{f'(r_H)g'(r_H)}}{4\pi}. \tag{2.9}$$

Several proposals were advanced to remedy the ambiguity of the temperature. In ref. [29], by analogy with a black body the emission rate is defined as the ratio of the amplitude square of the outgoing wave to that of the ingoing wave, which is given by S_-,

$$\frac{P_{\mathrm{out}}}{P_{\mathrm{in}}} = e^{-2\mathrm{Im}(S_+ - S_-)} = e^{-\omega/T_H}. \tag{2.10}$$

On the other hand, the isotropic coordinate and the proper distance along the radial direction are also employed to get the correct Hawking temperature in refs. [35, 36]. In fact, the ambiguity of the temperature is originated from the coordinate used to calculate the emission rate. Another ambiguity is that the action (2.5), in particular, the imaginary part is not invariant under canonical transformations [37, 38].

3. Tunneling in the Rindler spacetime

A Rindler spacetime is the spacetime covered by all time-like congruences of an accelerated particle. The Rindler spacetime has two horizons that separate the Minkowski spacetime into the right (R), the left (L), the past (P) and the future (F) wedges. The wedge (R) is causally disconnected from the wedge (L) and a timelike Killing vector normal to the spacelike Cauchy surface that covers both (R) and (L) defines a complete set of quantum fields. The thermal nature of Unruh effect comes from the fact that the physically accessible region (R) for an accelerated particle is causally disconnected with the other region (L) and the Minkowski vacuum defined in the union of (R) and (L) looks like a mixed state for the particle [3]. In this paper, following the arguments in refs. [2] and [3], we shall consider quantum tunneling from (L) to (R) wedge. However, in ref. [40] it was shown that the result of tunneling from (R) to (F) would differ from the Hawking temperature by a factor of two.

In two dimensions the right wedge (R) of the Rindler spacetime has the coordinate

$$
\begin{aligned}
t &= \rho_R \sinh(a\tau), \\
z &= \rho_R \cosh(a\tau),
\end{aligned}
\tag{3.1}
$$

with $\rho_R \geq 0$, and the left wedge (L) has

$$
\begin{aligned}
t &= \rho_L \sinh(a\tau), \\
z &= \rho_L \cosh(a\tau),
\end{aligned}
\tag{3.2}
$$

with $\rho_L \leq 0$. Here a is the acceleration of the particle. In both wedges the spacetime has the metric

$$
ds^2 = -(a\rho)^2 d\tau^2 + d\rho^2.
\tag{3.3}
$$

The right wedge is causally disconnected from the left wedge by horizons, $t = \pm z$, which correspond to $\rho = 0$. The accelerated particle would detect a thermal spectrum with the so-called Unruh temperature, $T_U = a/(2\pi)$, from the Minkowski vacuum [2]. From a view point of causality, the particle operators for the accelerated particle in (R) are expressed in terms of Minkowski operators in (R) and (L) through a Bogoliubov transformation, so the vacuum for the particle looks like a thermal vacuum in thermo-field dynamics [3].

However, quantum mechanically speaking, fields or particles can cross horizons with a certain probability. To cover both (R) and (L), we analytically continue the coordinate

$$
\rho_L = \rho_R e^{i\pi}.
\tag{3.4}
$$

The massive scalar field in (R) and (L) obeys the equation

$$
\left[-\frac{1}{(a\rho)^2} \frac{\partial^2}{\partial \tau^2} + \frac{\partial^2}{\partial \rho^2} - m^2 \right] \Phi(\tau, \rho) = 0,
\tag{3.5}
$$

and the spatial part, $\Phi = e^{-i\omega\tau}\varphi(\rho)$, satisfies the equation

$$\left[\frac{\partial^2}{\partial\rho^2} + \frac{\omega^2}{(a\rho)^2} - m^2\right]\varphi(\rho) = 0. \tag{3.6}$$

Note that eq. (3.6) is oscillatory for $|\rho| \leq \rho_c = \omega/(ma)$ and exponential otherwise. In quantum mechanics, it is a one-dimensional problem with the energy $-m^2$ and with the negative singular potential $-(\omega/a\rho)^2$.

For a tunneling wave function crossing $\rho = 0$, we may use the solution in (L)

$$\varphi_L(\rho) = \sqrt{\rho}J_{i\nu}(im\rho), \tag{3.7}$$

where $J_{i\nu}$ is the Bessel function with a complex order

$$\nu = \sqrt{\frac{\omega^2}{a^2} - \frac{1}{4}}. \tag{3.8}$$

The wave function in (R) tunneled from (L) may be found by analytically continuing the solution (3.7) via (3.4), which becomes

$$\begin{aligned}\varphi_R(\rho) &= (\rho_R e^{i\pi})^{1/2}J_{i\nu}(im\rho_R e^{i\pi}) \\ &= e^{-\nu\pi}e^{i\pi/2}(\rho_R)^{1/2}J_{i\nu}(im\rho_R),\end{aligned} \tag{3.9}$$

where the relation $J_\alpha(ze^{in\pi}) = e^{in\alpha\pi}J_\nu(z)$ for an integer n is used. Indeed, the solution (3.9) has an outgoing flux near the horizon in (R). Therefore we find the tunneling (emission) rate as the ratio of the amplitude square of the tunneled wave function in (R) from (L) to the amplitude square of the outgoing wave function $(\rho_R)^{1/2}J_{i\nu}(im\rho_R)$ with a given flux in (R):

$$P = e^{-2\nu\pi} \approx e^{-2\pi\omega/a}. \tag{3.10}$$

The tunneling rate is nothing but the Boltzmann factor with the Unruh temperature T_U. We further show that the tunneling rate, $P = e^{-\mathcal{S}_\omega}$, can also be obtained from the action, $\varphi = e^{iS(\rho)}$, where

$$\mathcal{S}_\omega = 2\,\mathrm{Im}S = -i\oint\sqrt{\frac{\omega^2}{(a\rho)^2} - m^2}d\rho = \frac{2\pi\omega}{a}, \tag{3.11}$$

with a contour enclosing $\rho = 0$.

We now compare the tunneling (emission) rate (3.11) in the Rindler coordinate with the pair production rate by an electric field in the Schwinger mechanism. In two dimensions the scalar field equation for charge e ($e > 0$) and mass m minimally coupled with the Coulomb gauge, $A_\mu = (-Ex, 0)$, takes the form

$$\left[\left(\frac{\partial}{\partial t} - ieEx\right)^2 - \frac{\partial^2}{\partial x^2} + m^2\right]\Phi = 0. \tag{3.12}$$

The spatial part, $\Phi = e^{-i\omega t}\varphi(x)$, satisfies

$$\left[\frac{\partial^2}{\partial x^2} + (\omega + eEx)^2 - m^2\right]\varphi(x) = 0. \tag{3.13}$$

In quantum mechanics, eq. (3.13) is a tunneling problem with energy $-m^2$ under the inverted harmonic potential. In refs. [46 – 48], the tunneling rate

$$P = e^{-\mathcal{S}} \tag{3.14}$$

is given by the WKB instanton action

$$\mathcal{S} = -i \oint \sqrt{(\omega + eEx)^2 - m^2}\, dx = \frac{\pi m^2}{eE}. \tag{3.15}$$

Here the contour integral is taken outside a contour in the complex x-plane.

We notice a similarity that the tunneling rate is given by the same formula

$$P = e^{-i\oint p}, \tag{3.16}$$

where for the Rindler case p is

$$p(\rho) = \sqrt{\frac{\omega^2}{(a\rho)^2} - m^2}, \tag{3.17}$$

while for the Schwinger mechanism

$$p(x) = \sqrt{(\omega + eEx)^2 - m^2}. \tag{3.18}$$

The contours include segments of real axis where p is real. However, for the Schwinger mechanism the contour integral is taken outside a contour that excludes a branch cut connecting two roots, $x_\pm = (-\omega \pm m)/(eE)$, while for the Rindler case the contour is taken inside a contour that excludes branch cuts from a root $\omega/(am)$ to the positive infinity and from another root $-\omega/(am)$ to the negative infinity.

4. Tunneling rate of black holes in the Rindler coordinate

From the argument in section 3 we shall use the Rindler coordinate of a black hole spacetime to calculate the tunneling rate. The spacetime region of the metric (2.1) near the event horizon may be locally approximated by a Rindler spacetime

$$ds^2 = -(\kappa\rho)^2 dt^2 + d\rho^2, \tag{4.1}$$

where the Rindler coordinate is used

$$\kappa\rho = \sqrt{f}, \quad \frac{dr}{d\rho} = \sqrt{g}. \tag{4.2}$$

From eq. (4.2) we find the surface gravity

$$\kappa = \frac{f'(r_H)}{2}\sqrt{\frac{g(r_H)}{f(r_H)}} = \frac{\sqrt{f'(r_H)g'(r_H)}}{2}.$$

(4.3)

The last equality holds only for non-extremal black holes. In the Rindler coordinate the instanton action is the contour integral in the complex ρ-plane

$$\mathcal{S}_\omega = -i \oint \sqrt{\frac{\omega^2}{f(\rho)} - m^2} d\rho = \frac{2\pi\omega}{\kappa}.$$

(4.4)

Here the contribution comes from the simple pole at $\rho = 0$, the event horizon. This leads to the emission rate for the Hawking radiation

$$P(\omega) = e^{-\mathcal{S}_\omega} = e^{-\omega/T_H}, \quad T_H = \frac{\kappa}{2\pi}.$$

(4.5)

As a by-product the contour integral is invariant under canonical transformations [38].

In the below we apply the emission rate in the Rindler coordinate to a Schwarzschild black hole, a non-extremal Reissner-Nordström black hole, a charged Kerr black hole, a de Sitter space and a Schwarzschild-anti de Sitter black hole.

4.1 Schwarzschild black hole

The Schwarzschild black hole with

$$f = g = 1 - \frac{2M}{r} = \frac{r - r_H}{r}, \quad (r_H = 2M)$$

(4.6)

has the metric

$$ds^2 = -(\kappa\rho)^2 dt^2 + \frac{(2r_H\kappa)^2}{(1-(\kappa\rho)^2)^4}d\rho^2,$$

(4.7)

in the Rindler coordinate

$$f = g = (\kappa\rho)^2.$$

(4.8)

Near the event horizon, $\rho \approx 0$ ($r \approx r_H$), the spacetime (4.7) approximately becomes a Rindler one with $\kappa = 1/(2r_H) = 1/(4M)$. Further, if the Euclidean time $\tau = it$ has a periodicity of $\tau = 2\pi/\kappa$, it is the Euclidean space without a deficit angle. In fact, κ is the surface gravity

$$\kappa = \frac{f'(r_H)}{2} = \frac{1}{4M},$$

(4.9)

and leads to the emission rate (4.5) with the Hawking temperature

$$T_H = \frac{1}{8\pi M}.$$

(4.10)

4.2 Reissner-Nordström black hole

The non-extremal Reissner-Nordström black hole with

$$f = g = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = \frac{(r - r_+)(r - r_-)}{r^2}, \tag{4.11}$$

has the event horizon, $r_+ = M + \sqrt{M^2 - Q^2}$, and the inner horizon, $r_- = M - \sqrt{M^2 - Q^2}$. With the coordinate

$$f = g = \frac{r_+^2}{r^2}(\kappa\rho)^2, \tag{4.12}$$

the metric becomes

$$ds^2 = -\frac{(2r_+)^2}{\left(r_+ + r_- + \sqrt{(r_+ - r_-)^2 + 4(\kappa r_+ \rho)^2}\right)^2}(\kappa\rho)^2 dt^2$$

$$+\frac{(\kappa r_+)^2 \left(r_+ + r_- + \sqrt{(r_+ - r_-)^2 + 4(\kappa r_+ \rho)^2}\right)^2}{(r_+ - r_-)^2 + 4(\kappa r_+ \rho)^2} d\rho^2. \tag{4.13}$$

Near the event horizon, $\rho \approx 0$, the spacetime (4.13) approximately becomes a Rindler one provided that $\kappa = (r_+ - r_-)/(2r_+^2)$, which is the surface gravity at the event horizon r_+,

$$\kappa = \frac{f'(r_+)}{2} = \frac{r_+ - r_-}{2r_+^2}. \tag{4.14}$$

The emission rate (4.5) is thus valid for non-extremal Reissner-Nordström black holes.

A caveat is that our emission rate relies on the Rindler coordinate. However, the extremal Reissner-Nordström black hole with $Q = M$

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dt^2 + \frac{dr^2}{\left(1 - \frac{M}{r}\right)^2} \tag{4.15}$$

cannot be approximated by the Rindler spacetime, since with $r = M + c\rho$ for any c, the metric has the form

$$ds^2 = -\frac{(c\rho)^2}{(M + c\rho)^2} dt^2 + \frac{(M + c\rho)^2}{\rho^2} d\rho^2. \tag{4.16}$$

Without the Rindler coordinate the instanton action (4.4) cannot be applied to extremal black holes.

4.3 Charged Kerr black hole

The charged Kerr black hole has the metric

$$ds^2 = -f dt^2 + \frac{dr^2}{g} + k(d\phi - \omega dt)^2 + \Sigma d\theta^2, \tag{4.17}$$

where

$$f = \frac{\Delta \Sigma}{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta},$$
$$g = \frac{\Delta}{\Sigma},$$
$$k = \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}{\Sigma},$$
$$\omega = \frac{a \sin^2 \theta (r^2 + a^2 - \Delta)}{(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta}, \tag{4.18}$$

where

$$\Delta = r^2 - 2Mr + a^2 + Q^2,$$
$$\Sigma = r^2 + a^2 \cos^2 \theta. \tag{4.19}$$

The event horizon is located at $r_+ = M + \sqrt{M^2 - a^2 - Q^2}$ and the inner horizon at $r_- = M - \sqrt{M^2 - a^2 - Q^2}$. With the coordinate

$$f = (\kappa x)^2, \tag{4.20}$$

the metric near the event horizon approximately takes the form

$$ds^2 = -(\kappa \rho)^2 dt^2 + \frac{4\kappa^2 (r_+^2 + a^2)^2}{(r_+ - r_-)^2} d\rho^2. \tag{4.21}$$

With the surface gravity

$$\kappa = \frac{r_+ - r_-}{2(r_+^2 + a^2)} = \frac{\sqrt{f'(r_+) g'(r_+)}}{2}, \tag{4.22}$$

the charged Kerr black hole can be written in the Rindler coordinate near the event horizon and has the emission rate (4.5).

4.4 De Sitter space

The de Sitter spacetime with

$$f = g = 1 - \frac{r^2}{l^2} = (\kappa \rho)^2 \tag{4.23}$$

has the event horizon at $r_H = l$. In the Rindler coordinate the de Sitter space becomes

$$ds^2 = -(\kappa \rho)^2 dt^2 + \frac{(l\kappa)^2}{1 - (\kappa \rho)^2} d\rho^2, \tag{4.24}$$

and locally a Rindler spacetime when $\kappa = 1/l$, which is the surface gravity. Thus the emission rate (4.5) is also valid for the de Sitter with the Hawking temperature

$$T_H = \frac{1}{2\pi l}. \tag{4.25}$$

4.5 Schwarzschild-anti de Sitter black hole

The Schwarzschild-anti de Sitter black hole with

$$f = g = 1 - \frac{2M}{r} + \frac{r^2}{l^2} \tag{4.26}$$

has the event horizon at

$$r_H = l \left[\left(\frac{M}{l} + \sqrt{\frac{1}{27} + (M/l)^2} \right)^{1/3} + \left(\frac{M}{l} - \sqrt{\frac{1}{27} + (M/l)^2} \right)^{1/3} \right]. \tag{4.27}$$

Near the event horizon, choosing the coordinate

$$f = g \approx (r - r_H) \times 2 \left(\frac{(r_H/l)^2 + Ml/r_H^2}{r_H} \right) = (\kappa\rho)^2 \tag{4.28}$$

we may write the metric approximately as

$$ds^2 \approx -(\kappa\rho)^2 + \frac{(l\kappa)^2}{(r_H/2 + Ml/r_H^2)^2} d\rho^2. \tag{4.29}$$

With the surface gravity

$$\kappa = \frac{1}{l} \left(\frac{r_H}{l} + \frac{Ml}{r_H^2} \right), \tag{4.30}$$

the metric becomes a Rindler spacetime. Thus the emission rate (4.5) is also valid for the Schwarzschild-anti de Sitter black hole.

5. Connection with other coordinates

In this section we discuss why the isotropic coordinate and the proper distance can recover the correct Hawking temperature and the Boltzmann factor [35, 36]. The spherically symmetric metric in eq. (2.1) can be written in the isotropic coordinate as

$$ds^2 = -f(\zeta)dt^2 + k(\zeta)(d\zeta^2 + \zeta^2 d\Omega^2), \tag{5.1}$$

where

$$\int \frac{d\zeta}{\zeta} = \int \frac{dr}{r\sqrt{g(r)}}. \tag{5.2}$$

For instance, the Schwarzschild black hole has the isotropic coordinate

$$ds^2 = -\left(\frac{1 - 2M/\zeta}{1 + 2M/\zeta} \right)^2 dt^2 + \left(\frac{1 + 2M/\zeta}{2} \right)^4 (d\zeta^2 + \zeta^2 d\Omega^2), \tag{5.3}$$

where $r = \zeta(1 + 2M/\zeta)^2/4$. The event horizon is located at $\zeta_H = r_H = 2M$. The metric near the event horizon is approximately given by

$$ds^2 \approx -\frac{(\zeta - 2M)^2}{(4M)^2} dt^2 + (d\zeta^2 + \zeta^2 d\Omega^2). \tag{5.4}$$

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Setting $(\zeta - 2M)/4M = \kappa\rho$ and $\kappa = 1/4M$, the Schwarzschild black hole metric becomes a Rindler one. This is the reason why the isotropic coordinate leads to the correct result.

Another coordinate is the proper distance

$$\sigma = \int \frac{dr}{\sqrt{g}}, \tag{5.5}$$

and its metric metric

$$ds^2 = -f(\sigma)dt^2 + d\sigma^2. \tag{5.6}$$

For non-extremal black holes, the leading terms are

$$f = f'(r_H)(r - r_H), \quad g = g'(r_H)(r - r_H). \tag{5.7}$$

Now the proper distance from the event horizon

$$\sigma = \frac{2}{\sqrt{g'(r_H)}}\sqrt{r - r_H}, \tag{5.8}$$

leads to

$$f = f'(r_H)\left(\frac{\sqrt{g'(r_H)}}{2}\right)^2 \sigma^2 = (\kappa\sigma)^2, \tag{5.9}$$

with the surface gravity (4.3). Therefore the proper distance method gives the same result as the Rindler coordinate.

6. Conclusion

We have studied the tunneling solution of a scalar field in the Rindler coordinate of a black hole spacetime. The tunneling solution in the black hole coordinate leads to a Boltzmann factor with a temperature twice of the Hawking temperature. However, using the analogy between the Schwinger mechanism in the Minkowski spacetime and the tunneling process in the Rindler spacetime of a black hole, we formulate the emission rate as a contour integral in both cases. This formulation avoids the coordinate-dependence of the emission rate in that it does not require the black hole coordinates first to express the emission rate and then to transform it to the Rindler coordinates. Only the surface gravity and local Rindler coordinates near horizons are needed for the calculation. In fact, our tunneling (emission) rate yields a Boltzmann factor with the correct Hawking temperature for non-extremal black holes such as a Schwarzschild black hole, a Reissner-Nordström black hole, a charged Kerr black hole, a de Sitter space, and a Schwarzschild-anti de Sitter black hole.

One of the reasons for using the Rindler coordinate near the event horizon is that the tunneling process of fields in the Rindler spacetime is analogous to the Schwinger process for pair production by an electric field. Also the field equation in the Rindler spacetime is quite similar to the equation minimally coupled with the gauge field of electric field. Indeed, the tunneling (emission) rate given by a contour integral takes the same form in the Rindler

coordinate of black holes for tunneling process and in the Minkowski for the Schwinger mechanism. Thus the tunneling idea of fields or particles crossing the horizon could be realized within the context of quantum field theory in a fixed spacetime background. Our emission rate in the Rindler coordinate has advantageous points: it resolves the controversy of coordinate-dependence of emission rate and the instanton action for the emission rate is indeed invariant under canonical transformations. A caveat, however, is that our formula cannot be applied to extremal black holes since they do not have Rindler coordinates near horizons.

Acknowledgments

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